



# 7

## TERMINOLOGY

abundant number  
amicable number  
closure law  
composite number  
continued fraction  
contradiction  
contraposition  
counter example  
deficient number  
double primes  
integer  
irrational number  
mathematical induction  
Mersenne prime  
natural number  
perfect number  
proper divisors  
rational number  
real number  
recurring decimal  
surd  
terminating decimal  
transcendental number  
twin primes

## REAL AND COMPLEX NUMBERS

# REAL NUMBERS AND PROOFS

- 7.01 Integers and subsets
- 7.02 Simple proofs involving integers
- 7.03 Decimal representation
- 7.04 Rational numbers
- 7.05 Irrational numbers
- 7.06 Real numbers
- 7.07 The principle of mathematical induction

Chapter summary

Chapter review



Prior learning


## PROOFS INVOLVING NUMBERS

- prove simple results involving numbers. (ACMSM061)

## RATIONAL AND IRRATIONAL NUMBERS

- express rational numbers as terminating or eventually recurring decimals and vice versa (ACMSM062)
- prove irrationality by contradiction for numbers such as  $\sqrt{2}$  and  $\log_2(5)$ . (ACMSM063)

## AN INTRODUCTION TO PROOF BY MATHEMATICAL INDUCTION

- understand the nature of inductive proof including the 'initial statement' and inductive step (ACMSM064)
- prove results for sums such as  $1 + 4 + 9 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$  for any positive integer  $n$  (ACMSM065)
- prove divisibility results, such as  $3^{2n+4} - 2^{2n}$  is divisible by 5 for any positive integer  $n$ . (ACMSM066) 

# 7.01 INTEGERS AND SUBSETS

Numbers were first used for counting. These counting numbers are now referred to as the **natural numbers**. They have the symbol  $N$ . So the set is  $N = \{1, 2, 3, 4, 5, \dots\}$ . They are also called the positive **integers**,  $Z^+$ . The set of integers,  $Z$  (or  $J$ ), contains the natural numbers, zero and the negative integers,  $Z^-$ . These integers make the set  $Z = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ .

The integers are divided into **even** and **odd numbers**. An even number is an integer that is divisible by 2. It is of the form  $n = 2k$ , where  $k$  is an integer.

The even numbers =  $\{\dots, -4, -2, 0, 2, 4, \dots\} = \{n: n = 2k, k \in Z\}$ .

An odd number is an integer that is not divisible by 2. The odd numbers are of the form  $n = 2k + 1$ , where  $k$  is an integer. When you divide an odd number by 2 you always get a remainder of 1.

The odd numbers =  $\{\dots, -3, -1, 1, 3, \dots\} = \{n: n = 2k + 1, k \in Z\}$ . Some people prefer the equivalent  $\{n: n = 2k - 1, k \in Z\}$ .

You can see patterns when operating with even and odd numbers. Some are shown in the table below.

Addition and Subtraction	Multiplication
even number $\pm$ even number = even number example: $2 + 6 = 8$	even number $\times$ even number = even number example: $4 \times 12 = 48$
odd number $\pm$ odd number = even number example: $3 + 9 = 12$	odd number $\times$ odd number = odd number example: $9 \times 11 = 99$
even number $\pm$ odd number = odd number example: $12 - 3 = 9$	even number $\times$ odd number = even number example: $10 \times 9 = 90$

## ○ Example 1

- a Show that when an odd number is subtracted from an even number, the result is an odd number.  
 b Show that when two odd numbers are multiplied together, the result is an odd number.

### Solution

- a Write an odd number.  
 Write an even number.  
 Subtract the numbers.  
 Substitute.

$$\text{Let } x = 2a + 1, \text{ where } a \in \mathbb{Z}$$

$$\text{Let } y = 2b, \text{ where } b \in \mathbb{Z}$$

$$y - x \\ = 2b - (2a + 1)$$

Remove the brackets.

$$= 2b - 2a - 1$$

Factorise  $2b - 2a$ .

$$= 2(b - a) - 1$$

State the nature of  $b - a$ .

$b - a$  is an integer as  $a$  and  $b$  are integers

State the nature of  $2(b - a)$ .

$2(b - a)$  is even as it is  $2c$ , where  $c \in \mathbb{Z}$

Write the conclusion.

Hence  $2(b - a) - 1$  is odd. **QED**

- b Represent one of the odd numbers as  $2a + 1$  and the other as  $2b + 1$ , where  $a$  and  $b$  are integers.

Multiply the two numbers together.

$$(2a + 1)(2b + 1)$$

Expand the brackets.

$$= 4ab + 2a + 2b + 1$$

Factorise  $4ab + 2a + 2b$ .

$$= 2(2ab + a + b) + 1$$

Show that the result is odd.

$2ab + a + b$  is an integer as  $a$  and  $b$  are integers.

$2(2ab + a + b)$  is even.

Hence  $2(2ab + a + b) + 1$  is odd. **QED**

A **perfect number** is a positive integer that is equal to the sum of its positive **proper divisors**. Proper divisors are the factors of the number, excluding the number itself. 6 and 28 are perfect numbers because  $1 + 2 + 3 = 6$  and  $1 + 2 + 4 + 7 + 14 = 28$ . The Greek mathematician Nicomachus described **deficient numbers** as numbers where the sum of the proper divisors is less than the number; for example, 8 is deficient as  $1 + 2 + 4 < 8$ . He called numbers where the sum of the proper divisors is greater than the number **abundant numbers**. 12 is the first abundant number. The sum of its proper factors is  $1 + 2 + 3 + 4 + 6 > 12$ .

**Amicable numbers** are pairs of numbers where each one is the sum of the proper divisors of the other. The first pair of amicable numbers is 220 and 284.

The proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110.

$$1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284$$

The proper divisors of 284 are 1, 2, 4, 71 and 142.

$$1 + 2 + 4 + 71 + 142 = 220$$

Their divisors add to each other, so they *are* amicable numbers.

Proper divisors are also called **proper factors**. The smallest prime number is 2 and it is the only even prime number. There are 25 prime numbers that are less than 100: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89 and 97.

## IMPORTANT

A **prime number** is a positive integer that has exactly two factors, one and itself. A **composite number** is a positive integer that has more than two factors.

### ○ Example 2

- a Show that 496 is a perfect number.  
 b Write 496 in the form  $2^{p-1}(2^p - 1)$ , where  $p$  is a prime number.  
 c Is  $2^6(2^7 - 1)$  a perfect number?

#### Solution

- a List the proper divisors of 496. 1, 2, 4, 8, 16, 31, 62, 124, 248  
 Add the proper divisors.  $1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248 = 496$   
 Write the conclusion. Hence 496 is a perfect number.
- b Test some prime numbers.  $2^4(2^5 - 1) = 496$   
 Try  $p = 5$ .  
 Note that  $2^2(2^3 - 1) = 28$  and  
 $2^1(2^2 - 1) = 6$ .
- c List the proper divisors of  $2^6(2^7 - 1)$ .  $2^6(2^7 - 1) = 8128$   
1, 2, 4, 8, 16, 32, 64, 127, 254, 508, 1016, 2032, 4064  
 Add the proper divisors.  $1 + 2 + 4 + 8 + 16 + 32 + 64 + 127 + 254 + 508$   
 $+ 1016 + 2032 + 4064 = 8128$   
 Write the answer.  $2^6(2^7 - 1)$  is a perfect number.

Notice in Example 2 that the proper factors of 496 are 1, 2, ...,  $2^4$  and  $31, 2 \times 31, \dots, 2^3 \times 31$ .

To check if a number is prime, you only need to check if the number is divisible by the prime numbers less than or equal to its square root. For example, to check if 59 is prime, you only need to check the prime numbers up to  $\sqrt{59} \approx 7.68$ . Since 59 is not divisible by 2, 3, 5 or 7, it must be prime.

#### Proof

Suppose that the number's square root is a whole number. It follows immediately that the number is not prime. Consider cases where the square root is not a whole number.

Suppose there was a number whose only factors were bigger than the square root.

Let the number be  $n$  and the factor larger than the square root be  $f$ .

Since  $f$  is a factor,  $fp = n$  for some positive integer  $p$ .

It follows that  $p = \frac{n}{f}$ .

But  $f > \sqrt{n}$ , so  $\frac{1}{f} < \frac{1}{\sqrt{n}}$  and thus  $p = \frac{n}{f} < \frac{n}{\sqrt{n}} = \sqrt{n}$ .

Thus, the number has a factor less than  $\sqrt{n}$ .

This contradicts the assumption. Thus there is no number whose only factors are bigger than its square root.

We've also shown that any factor larger than  $\sqrt{n}$  has a co-factor less than  $\sqrt{n}$ .

Thus, it is sufficient to check for factors that are less than or equal to the square root. **QED**

As an example, consider  $n = 103$ ,  $\sqrt{103} \approx 10.1$ . You only need to check if 103 is divisible by 2, 3, 5 and 7. 103 is not divisible by 2 as the last digit is not even, nor 3 as the sum of the digits, which is 4, is not divisible by 3, nor 5 as the last digit is neither 0 or 5 and not 7. Hence 103 is prime.

The **Sieve of Eratosthenes** is a method for finding prime numbers up to a particular number. The table below shows how it works for 50. Circle the lowest prime number, 2 and cross out all the multiples of that number, 4, 6, 8, ... Select the next lowest prime number, 3 and cross out all the multiples of that number that have not already been crossed out, which means you can start at  $3^2 = 9$  as the smaller multiples have been crossed out. Continue the process until you have checked the prime numbers less than or equal to  $\sqrt{n}$ . In this case  $\sqrt{50} \approx 7.07$ . So cross out the multiples of the prime numbers up to and including 7. The rest of the unmarked numbers should be prime.

	②	③	4	⑤	6	⑦	8	9	10
11	<del>12</del>	13	<del>14</del>	<del>15</del>	<del>16</del>	17	<del>18</del>	19	<del>20</del>
<del>21</del>	<del>22</del>	23	<del>24</del>	<del>25</del>	<del>26</del>	<del>27</del>	<del>28</del>	29	<del>30</del>
31	<del>32</del>	<del>33</del>	<del>34</del>	<del>35</del>	<del>36</del>	37	<del>38</del>	<del>39</del>	<del>40</del>
41	<del>42</del>	43	<del>44</del>	<del>45</del>	<del>46</del>	47	<del>48</del>	<del>49</del>	<del>50</del>

Hence 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43 and 47 are prime numbers.

There are many special types of prime numbers. **Double or twin primes**, for example, differ by two. Three and five are twin primes and five and seven are twin primes. Another type of prime number is a **Mersenne prime**. A Mersenne prime is a (prime) number that can be written in the form  $2^p - 1$ , where  $p$  is prime. For example,  $2^2 - 1 = 3$ ,  $2^3 - 1 = 7$ ,  $2^5 - 1 = 31$  and  $2^7 - 1 = 127$  are Mersenne primes, but  $2^4 - 1 = 15$  is not as 4 is not a prime number.

### ○ Example 3

Is  $2^{11} - 1 = 2047$  a Mersenne prime?

#### Solution

Find  $\sqrt{2047}$ .

$$\sqrt{2047} \approx 45$$

Write the prime numbers less than  $\sqrt{2047}$ .

Check 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41 and 43

Perform the divisibility tests.

2047 is not divisible by 2 as the last digit is not even, nor 3 as the sum of the digits, 13, is not divisible by 3. It is not divisible by 5 as the last digit is not 0 or 5.

Check the rest on a calculator.

For example,  $\frac{2047}{7} = 292.4\dots$

It is not divisible by 7, 11, 13, 17 and 19 either.

$\frac{2047}{23} = 89$ , so it is divisible by 23.

2047 is not prime. Hence  $2^{11} - 1$  is not a Mersenne prime.

Write the answer.

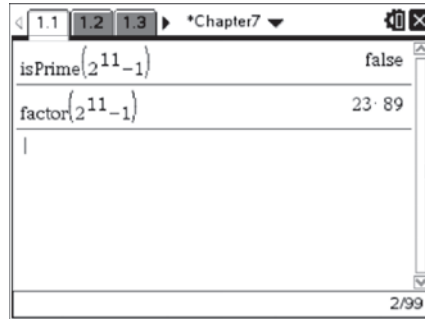
You can easily check if a number is prime using technology.

### Ti-Nspire CAS

Insert a calculator page, and type the commands directly or press  $\left[\frac{\square}{\square}\right]$  and select **isPrime** (. Complete the command and press  $\left[\text{enter}\right]$ .

Use  $\left[\text{menu}\right]$  and select 3: Algebra and 2: Factor to find the factors.

**isPrime** tells you if a number is prime.

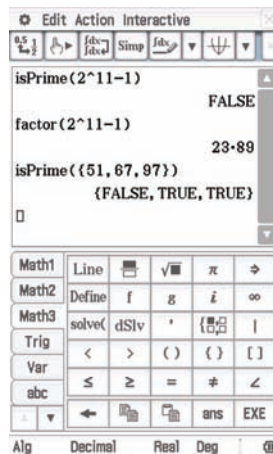


### ClassPad

Use the  $\sqrt[n]{a}$  application. Navigate to the list of functions (press  $\left[\text{Keyboard}\right]$  and tap  $\left[\downarrow\right]$ ) and find **isPrime**( in the list. The command **isPrime** tells you if a number is prime or not.

The command **factor** gives you the number as a product using its prime factors.

**isPrime** can be used to test more than one number at a time by entering the values inside of a pair of braces with each value separated from the previous one by a comma.



## INVESTIGATION Playing with integers

### 1 Finding primes

- a Answer the following questions by downloading and using the Sieve of Eratosthenes at the site given using the web link.

Note: To run the demonstration you need to have Mathematica or install the free Mathematica player from Wolfram.

- i How many prime numbers are less than 315?
- ii What was the largest prime that needed to be tested?
- iii What is the largest prime that needs to be tested to determine the number of primes less than 960?



Sieve of Eratosthenes

**b** Use a suitable CAS calculator, Mathematica or the website to answer the following questions.

- i** List the primes less than 100.
- ii** Find the 1000th smallest prime number.
- iii** How many prime numbers are there between 7000 and 8000?
- iv** Is 10 347 a prime number?



**2** Perfect numbers and Mersenne primes

View the demonstration at the website.

- a i** How many deficient numbers are less than 6?
- ii** How many abundant numbers are less than 28?
- iii** Is 496 a perfect number?
- b** Using a suitable CAS, Mathematica or the website:
  - i** show that 496 and 8128 are perfect numbers
  - ii** write 496 and 8128 as products of their prime factors
- c** Show, by summing the factors of a perfect number, that if a perfect number can be written in the form  $2^{p-1} \times b$ , where  $b$  is prime, then  $b = 2^p - 1$ .

A Mersenne prime is a prime number which can be written in the form  $2^p - 1$ , where  $p$  is a prime number.

**d** Use a suitable CAS application such as Mathematica to complete the following table.

$p$	Is $2^p - 1$ prime?	Is $2^{p-1}(2^p - 1)$ a Perfect number?
2	Yes	Yes
3		
5		
7		
11		
13		
17		
19		
23		
29		
31		

- e i** Is a perfect number ever a prime number? Explain.
- ii** What conclusion can you draw from the table?
- f** How many perfect numbers were found by October 2009?





## EXERCISE 7.01 Integers and subsets



Proper  
divisors

### Concepts and techniques

- 3 belongs to which one of the following sets?  
A counting numbers      B  $\mathbb{Z}^+$       C  $\mathbb{N}$       D odd numbers  
E even numbers
- Example 1** a Show that when two odd numbers are added together, the result is an even number.  
b Show that when an odd number is multiplied by an even number, the result is an even number.
- Which one of the following is not a proper divisor of 312?  
A 1      B 13      C 24      D 104      E 312
- List the numbers less than 20 which are abundant.
- Example 2** a Show that 33 550 336 is a perfect number.  
b Write 33 550 336 in the form  $2^p - 1(2^p - 1)$ , where  $p$  is a prime number.
- Show that 1184 and 1210 are amicable numbers.
- Is the set of prime numbers a subset of the set of odd numbers? Explain.
- Use the Sieve of Eratosthenes to find the prime numbers less than 100.
- Determine whether each of the following numbers is prime.  
a 247      b 631
- Example 3** Is  $2^{13} - 1$  a Mersenne prime?
- If  $(a, b)$  represents twin prime pairs, how many pairs exist if  $a$  and  $b$  are less than 50?

### Reasoning and communication

- a  $2^{17} - 1$  is a prime number. What should it be multiplied by to get a perfect number?  
b Explain why  $2^a(2^{23} - 1)$ , where  $a$  is a positive integer, cannot represent a perfect number.

## 7.02 SIMPLE PROOFS INVOLVING INTEGERS

A mathematical proof is a logical argument which demonstrates that something is true. There are several different types of proofs. **Direct proofs** were used in the previous section to show that, for example, if you add an even and an odd integer, the result is an odd integer. Other types of proof include proof by **counter example**, **contraposition** and **contradiction**. You have already seen a proof by contradiction. This was used in the previous section to show that if a number is prime, you only need to check if the number is divisible by the prime numbers less than or equal to its square root.

**Closure laws** are statements about operations. A set is closed under an operation if the result of the operation on members of the set always belongs to the set.

## IMPORTANT

### Closure law of addition

The sum of any two integers results in another integer.

In set notation,  $a, b \in \mathbf{Z} \Rightarrow c = a + b \in \mathbf{Z}$ .

### Closure law of multiplication

The product of any two integers results in another integer.

In set notation,  $a, b \in \mathbf{Z} \Rightarrow c = a \times b \in \mathbf{Z}$ .

A formal definition of closure using mathematical symbols is given below. The symbol  $*$  is often used to denote an operation using two numbers, called a binary operation. Remember that  $\Leftrightarrow$  means ‘if and only if’ and  $\forall$  means ‘for all’.

To prove whether a set is closed or not, you can do a direct proof or find a counterexample to show that it is not closed. Both are shown in Example 4.

A **counterexample** is where you give an example to show that a statement is false. It is only necessary to give one example. For example, the statement ‘all months of the year have 31 days’ is false because the counterexample September has 30 days.

## IMPORTANT

A set  $S$  is **closed** under an operation  $*$  if and only if, for all  $a$  and  $b$  in  $S$ ,  $a * b$  is also in  $S$ .

In set notation:

$S$  is closed under  $*$   $\Leftrightarrow a * b \in S \forall a, b \in S$ .

## ○ Example 4

Consider the set  $T = \{1, 3, 5, 7, 9, \dots\}$ .

- a Give a counterexample to show that each of the following statements is false.
  - i The set is closed under addition.
  - ii The set is closed under subtraction.
- b Using a direct proof, show a positive odd number is never produced by addition of two members of the set.

### Solution

- a i Give a counterexample.

Write the conclusion.

$$1 + 3 = 4$$

Since 4 is not an odd number, the set  $T$  is not closed under addition. **QED**

- ii Give a counterexample.

Write the conclusion.

$$1 - 3 = -2$$

Since  $-2$  is not an odd number, the set  $T$  is not closed under subtraction. **QED**

b Choose odd numbers.	Let $x = 2a + 1$ and $y = 2b + 1$
Add the numbers.	$x + y$
Substitute expressions.	$= 2a + 1 + 2b + 1$
Put it in the form of an even integer.	$= 2a + 2b + 2$ $= 2(a + b + 1)$
Write the conclusion.	$2(a + b + 1)$ is even so addition of two numbers from the set $T$ never gives a member of the set.

**QED**

Another type of proof is proof by contraposition. Some examples are:

- saying that ‘if you wear Ugg boots you will have warm feet’ is the same as saying ‘if you do not have warm feet you cannot be wearing Ugg boots’
- saying that ‘a dog is an animal with four legs’ is the same as saying that ‘an animal which doesn’t have four legs cannot be a dog’

### IMPORTANT

#### Contraposition

The statement ‘If  $p$  then  $q$ ’ is equivalent to ‘If not  $q$  then not  $p$ ’

You can prove that ‘if a number is divisible by 6, then it is divisible by 3’ by proving the contrapositive statement ‘if a number is not divisible by 3, then it is not divisible by 6’.

### IMPORTANT

The phrase ‘if and only if’, shown by the symbol  $\Leftrightarrow$  means that both the first statement and its converse are true.

It means that the statement is two-way.

### ○ Example 5

- a Prove that if  $n^2$  is odd, then  $n$  is odd, where  $n \in \mathbf{Z}$ .  
 b Prove that  $n^2$  is odd if and only if  $n$  is odd, where  $n \in \mathbf{Z}$ .

#### Solution

- |                                     |   |
|-------------------------------------|---|
| a Write a contrapositive statement. | If $n$ is even, then $n^2$ is even, where $n \in \mathbf{Z}$ .  |
| Assume that $n$ is an even integer. | Let $n = 2k$ , where $k$ is an integer.   |
| Square both sides of the equation.  | $n^2 = 4k^2 = 2(2k^2)$  |
| Write the answer.                   | Since $k$ is an integer, $2k^2$ is an integer and $2(2k^2)$ is an even integer. That is, it cannot be odd.<br>Hence if $n^2$ is odd, then $n$ is odd, where $n \in \mathbf{Z}$ . <b>QED</b> |

b One half has been proven.	Only the converse is needed.
Assume that $n$ is odd.	Let $n = 2k + 1$ , where $k \in \mathbf{Z}$ , be an odd integer.
Square $n$ .	$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$
What kind of number is $n^2$ ?	But $2(2k^2 + 2k) + 1$ is odd, so $n^2$ is odd
State the result.	From part a, if $n^2$ is odd then $n$ is odd, and from the above, if $n$ is odd then $n^2$ is odd, so $n^2$ is odd if and only if $n$ is odd, where $n \in \mathbf{Z}$ . <b>QED</b>

To prove a statement  $A$  by contradiction, assume that  $A$  is false or  $A'$  is true. Show that  $A'$  leads to  $B$  and  $B$  contradicts  $A'$ . Hence  $A'$  is false and  $A$  is true. The proof in Example 6 is due to Euclid who lived around 300 BC.

### ○ Example 6

Show that there are infinitely many prime numbers.

#### Solution

Write the contradictory statement.	There are a finite number of prime numbers.
Use mathematical notation.	Let the $n$ primes be $p_1, p_2, p_3, \dots, p_n$ .
Choose a particular number $a$ which is bigger than any of the primes.	$a = p_1 \times p_2 \times p_3 \times \dots \times p_n + 1$ $a$ cannot be prime as it is bigger than all the primes.
Write the steps which show that there is a contradiction.	If $a$ is not prime, it must be divisible by one of the primes. However, when you divide $a$ by any of the primes, there is always a remainder of 1. Hence $a$ must be prime or have new prime factors.
State the answer.	This is a contradiction. So the original assumption is incorrect and there are infinitely many primes. <b>QED</b>

## EXERCISE 7.02 Simple proofs involving integers

### Concepts and techniques

- Example 4** Consider the set  $O = \{1, 3, 5, 7, 9, \dots\}$ .
  - Provide a counter example to show that the set is not closed under division.
  - Use a direct proof to show that the set is closed under multiplication.
- Is the set of negative integers closed under
    - addition?
    - subtraction?
    - multiplication?
    - division?
  - Give a counter example for the ones that are not closed.
- Is the set of integers closed under division? If not, give a counter example.

- 4 Suppose that  $A = \{-1, 0, 1\}$ .
  - a Is  $A$  closed under addition?
  - b Is  $A$  closed under multiplication?
- 5 Give counterexamples to show that each of the following are false.
  - a All prime numbers are odd.
  - b When you square an integer, the result is always a positive integer.
- 6 Show that the sum of two consecutive odd integers is divisible by 4.
- 7 Prove that every odd integer is the difference of two perfect squares.
- 8 Show that if  $p$  and  $q$  are twin primes, then 1 more than their product is divisible by their average.

## Reasoning and communication

Use proof by contraposition to prove questions 9 to 12.

- 9 **Example 5** a Prove that if  $n^2$  is even, then  $n$  is even, where  $n \in \mathbf{Z}$ .  
b Prove that  $n^2$  is even if and only if  $n$  is even, where  $n \in \mathbf{Z}$ .
- 10 a If for two integers  $p$  and  $q$ ,  $p + q$  is even, then  $p$  and  $q$  have the same parity (both odd or both even).  
b Prove that the sum of two integers,  $p$  and  $q$ , is even if and only if  $p$  and  $q$  have the same parity.
- 11 If for two integers  $p$  and  $q$ ,  $pq$  is even, then both of them will not be odd.
- 12 If for two integers  $p$  and  $q$ ,  $pq$  is odd, then  $p$  and  $q$  must be odd.

Use proof by contradiction to prove questions 13 and 14.

- 13 **Example 6** A Diophantine equation is one where the solutions are required to be integers. Prove by contradiction that there are no positive integer solutions to the Diophantine equation  $a^2 - b^2 = 1$ .
- 14 Prove by contradiction that there are an infinite number of even integers.
- 15 Prove that the only prime of the form  $n^3 - 1$  is 7.
- 16 Given  $x \in \mathbf{Z}$ , prove that if  $x^2 - 2x + 5$  is odd, then  $x$  is even.

## 7.03 DECIMAL REPRESENTATION

**Terminating** decimals are decimals which have a finite number of digits after the decimal place like 0.25. **Recurring** decimals are decimals that have digits in a repeating pattern after the decimal place; for example,  $\frac{1}{7} = 0.142\ 857\ 142\ 857\ 142\ 857\dots = 0.\overline{142\ 857}$  or  $0.\dot{1}42\ 85\dot{7}$ . The sequence of digits that is repeated is 142857. You put a **vinculum** (line or bar) above these numbers to show that they are repeated or a dot above the first and last digits of the repeating part.

Terminating and recurring decimals can all be expressed in the form  $\frac{a}{b}$ , where  $a$  and  $b$  are integers and  $b$  is not equal to zero.  $\frac{a}{b}$  is called a **fraction**. We call  $a$  the **numerator** and  $b$  the **denominator**.

The line above the denominator is also called a vinculum. Fractions are said to be written in simplest form when  $x = \frac{c}{d}$ , and  $c$  and  $d$  have no common factors except 1, and  $d$  does not equal zero.

In this section you will be converting terminating and recurring decimals to fractions and vice versa.

The following example shows this.

### ○ Example 7

a Convert the following terminating decimals to fractions.

- i 9.9999                      ii -435.078 65

b Convert the following fractions to terminating decimals.

- i  $-\frac{6666}{10\ 000}$                       ii  $\frac{237}{8}$

### Solution

a i There are four digits after the decimal place.

So multiply 9.9999 by 10 000 and divide by 10 000.

Write the answer.

$$9.9999 = \frac{9.9999 \times 10\ 000}{10\ 000} = \frac{99\ 999}{10\ 000}$$

ii There are five digits after the decimal place.

So multiply -435.078 65 by 100 000 and divide by 100 000.

Write the answer.

Answers are normally written in simplest form.

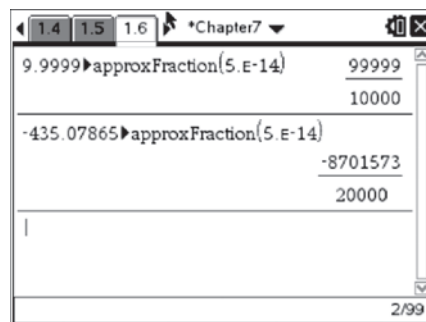
$$-435.078\ 65 \times \frac{100\ 000}{100\ 000}$$

$$= -\frac{43\ 507\ 865}{100\ 000}$$

$$= -\frac{8\ 701\ 573}{20\ 000}$$

### TI-Nspire CAS

Insert a calculator page. Type the number and select **menu** 2: Number and 2: Approximate to Fraction.



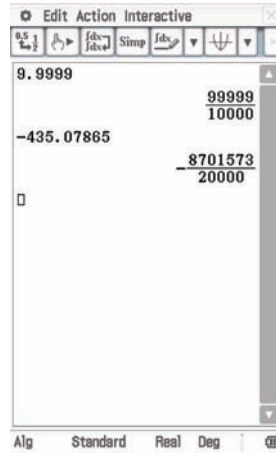
### ClassPad

Use the  $\sqrt{x}$  application.

Set the calculator to **Standard**.

Simply enter the decimal and press **[EXE]** and the calculator will convert it to a fraction, expressed in simplest form.

When entering negative numbers, use the **[(-)]** key.



b i Divide  $-6666$  by  $10\,000$ .

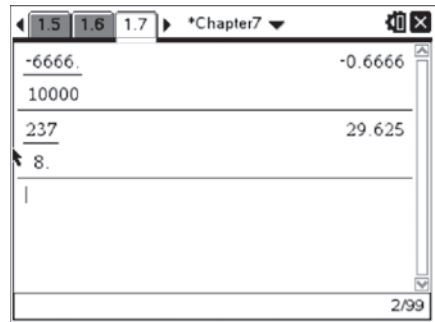
ii Divide  $237$  by  $8$ .

$-0.6666$

$29.625$

### TI-Nspire CAS

Insert a calculator page. Press **[(-)]** and type the number. Press **[ctrl]** and **[enter]** to get the decimal equivalent.



### ClassPad

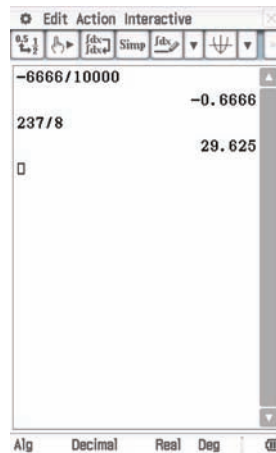
Use the  $\sqrt{x}$  application.

Set the calculator to **Decimal**.

Enter the expression and press **[EXE]**.

If you want the answer as a fraction, just set the calculator to Standard.

When entering negative numbers, use the **[(-)]** key.



The next example shows how to convert a fraction to a recurring decimal. You keep dividing until a pattern emerges. The pattern will be a finite sequence of digits that keep recurring.

## ○ Example 8

Convert each of the following to a recurring decimal.

a  $\frac{239}{11}$

b  $\frac{239}{12}$

### Solution

a Divide 239 by 11.

Stop when a pattern emerges.

There are two recurring digits.

Put a line above the recurring digits.

$$\begin{array}{r} 21.7272\dots \\ 11 \overline{)239.803030} \\ \underline{2172} \phantom{0} \\ 190 \phantom{0} \\ \underline{198} \phantom{0} \\ 200 \phantom{0} \\ \underline{198} \phantom{0} \\ 200 \phantom{0} \\ \underline{198} \phantom{0} \\ 200 \phantom{0} \\ \underline{198} \phantom{0} \\ 200 \phantom{0} \end{array}$$

b Divide 239 by 12.

Stop when a pattern emerges.

There is one recurring digit.

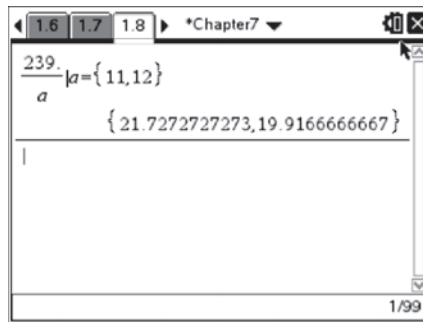
Put a bar above the recurring digit.

$$\begin{array}{r} 19.9166\dots \\ 12 \overline{)239.102080} \\ \underline{240} \phantom{0} \\ \phantom{2}00 \phantom{0} \\ \underline{192} \phantom{0} \\ 80 \phantom{0} \\ \underline{84} \phantom{0} \\ \phantom{8}0 \phantom{0} \\ \underline{84} \phantom{0} \\ \phantom{8}0 \phantom{0} \\ \underline{84} \phantom{0} \\ \phantom{8}0 \phantom{0} \end{array}$$

### TI-Nspire CAS

Insert a calculator page. Type the number and press  $\text{ctrl}$ ,  $\text{=}$  and select  $|$ . Press  $\text{ctrl}$ ,  $\text{}$ , and type the remaining numbers.

You might need to change the Document Settings to get a 12-digit display.

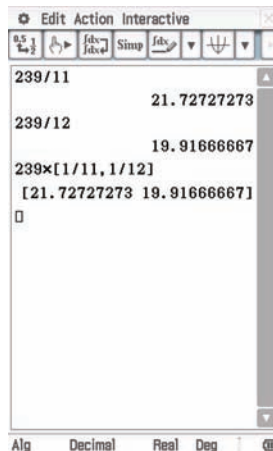


### ClassPad

Main  $\sqrt{x}$  application.

Set the calculator to **Decimal**.

Enter the expression and press  $\text{EXE}$ .





There are a number of ways to convert recurring decimals to fractions. The method shown in Example 12 is summarised in the box below.

## IMPORTANT

### Converting a recurring decimal to a fraction

- 1 Let  $x$  represent the number.
- 2 Multiply the number  $x$  by  $10^n$  where  $n$  is the number of recurring digits.
- 3 Subtract 1 from 2.
- 4 Rearrange the equation to write  $x$  as a fraction.
- 5 Write your answer in simplest form.

### ○ Example 9

Express the following recurring decimals in rational form.

- a  $0.\overline{9}$       b  $0.\overline{36}$       c  $2.\overline{7451}$

#### Solution

- a** Write the original number as  $x$ , showing the recurring pattern.  $x = 0.9999\dots$
- Since one digit recurs, multiply  $x$  by 10.  $10x = 9.9999\dots$
- Subtract  $x$  from  $10x$ . 

$$\begin{array}{r} 10x = 9.9999\dots \\ - x = 0.9999\dots \\ \hline 9x = 9 \end{array}$$
- Divide by 9.  $x = \frac{9}{9} = 1$
- b** Write the original number as  $x$ , showing the recurring pattern.  $x = 0.36\ 36\ 36\ 36\ 36\ \dots$
- Since two digits recur, multiply by 100.  $100x = 36.36\ 36\ 36\ 36\ \dots$
- Subtract  $x$  from  $100x$ . 

$$\begin{array}{r} 100x = 36.36\ 36\ 36\ 36\ \dots \\ - x = 0.36\ 36\ 36\ 36\ \dots \\ \hline 99x = 36 \end{array}$$
- Divide by 99 and simplify. 

$$\begin{aligned} x &= \frac{36}{99} \\ &= \frac{4}{11} \end{aligned}$$
- c** Write the original number as  $x$ , showing the recurring pattern.  $x = 2.7\ 451\ 451\ 451\ 451\ \dots$
- Since three digits recur, multiply by 1000.  $1000x = 2745.145\ 145\ 145\ 1\dots$

Subtract  $x$  from  $1000x$ .

$$\begin{array}{r} 1000x = 2745.1\ 451\ 451\ 451\ 451\ \dots \\ - x = \quad 2.7\ 451\ 451\ 451\ 451\ \dots \\ \hline 999x = 27\ 42.4 \end{array}$$

Multiply by 10.

$$9990x = 27\ 424$$

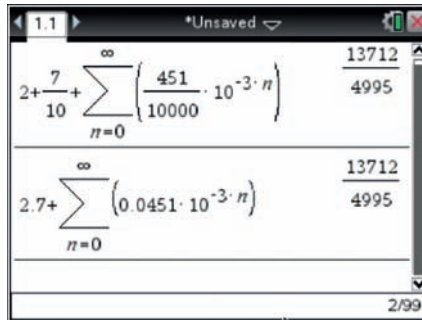
Divide by 9990 and simplify.

$$\begin{aligned} x &= \frac{27\ 424}{9990} \\ &= \frac{13\ 712}{4995} \end{aligned}$$

### TI-Nspire CAS

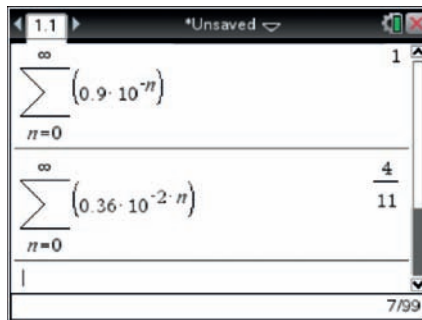
$$2.\overline{7451} = 2.745\ 145\ 145\ 145\ \dots$$

You can enter this as shown on the top line, using the  $\frac{\square}{\square}$  key to get the fraction and  $\Sigma$  templates and the  $\infty$  key to get  $\infty$ . Otherwise, change the Calculation Mode ( $\text{doc}$ , 7: Settings and Status, 2: Document settings) to Exact and enter as decimals as shown.



You can also do this for parts **a** and **b** but the whole expression is repeating for these, so you don't need to put anything before  $\Sigma$ .

$0.999\dots$  repeats one digit only so you use  $10^{-1n}$ , while  $0.3636\dots$  repeats 2 digits so you use  $10^{-2n}$ .



### ClassPad

$$2.745\ 145\ 145\ 145\ 1\dots$$

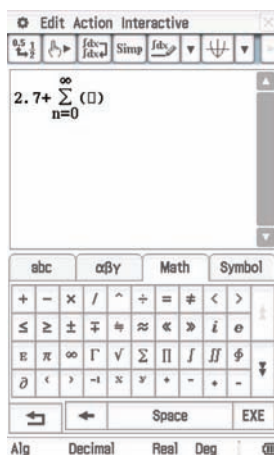
Use the  $\sqrt{\square}$  application and set the calculator to **Standard**.

Enter the non-recurring part, if it exists.

In this case it is 2.7.

Press  $\frac{\square}{\square}$  and tap the symbol  $\Sigma$ , found after pressing **Keyboard** and tapping **Math2**.

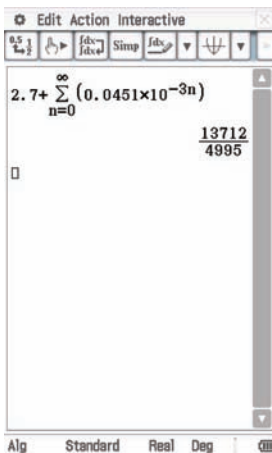
Fill it in to make  $\sum_{n=0}^{\infty} (\square)$ .



In the bracket, insert the exact value of the first recurring part, 0.0451.

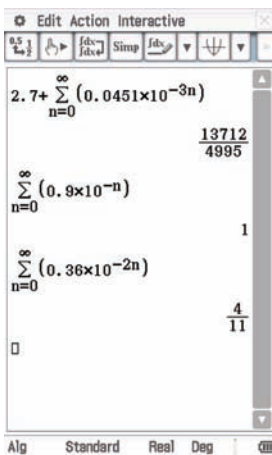
There are 3 repeating digits, so multiply by 10 to the power  $-3n$ .

Tap  $\boxed{EXE}$  or press  $\boxed{EXE}$ .



This can also be done for parts **a** and **b**, 0.9999... and 0.3636...

In these cases, the whole expression is repeating.



The result in Example 12a,  $0.\overline{9} = 1$  is significant. This means that numbers such as  $3.999\dots$  and 4 are equivalent. It is also very clear that every rational number can be expressed in many different ways; for example,  $-2\frac{2}{5} = -2.4 = -2.400\dots = -2.399\dots = -\frac{12}{5}$ .

## EXERCISE 7.03 Decimal representation



### Concepts and techniques

- Example 7** a Convert the following terminating decimals to fractions.

i 3.245                                  ii  $-521.078574$

b Convert the following fractions to terminating decimals.

i  $\frac{5659}{8}$                                   ii  $\frac{31129}{500}$
- Which one of the following can be written as a terminating decimal?

A  $\frac{189\ 767}{3}$                           B  $\frac{189\ 767}{7}$                           C  $\frac{189\ 767}{8}$                           D  $\frac{189\ 767}{9}$                           E  $\frac{189\ 767}{11}$
- Example 8** Convert each of the following to a recurring decimal.

a  $\frac{564}{11}$                                   b  $\frac{239}{13}$                                   c  $\frac{22}{7}$

4 **Example 9** Convert each of the following recurring decimals to fractions.

- a  $0.\bar{1}$                       b  $0.58\bar{3}$                       c  $3.1\bar{28}$                       d  $1.\bar{714\ 285}$                       e  $1.4\bar{9}$

## Reasoning and communication

- 5 Recurring decimals can be expressed as the sum of an infinite geometric series.  $S_{\infty} = \frac{a}{1-r}$ , where  $a$  is the first term and  $r$  is the common ratio, i.e., the second term divided by the first term.  
 Given  $\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \dots$ ,
- write the sum as a recurring decimal
  - write the sum as a fraction using  $S_{\infty} = \frac{a}{1-r}$
- 6 Consider  $\frac{1}{7}$ ,  $\frac{1}{17}$  and  $\frac{1}{19}$ .
- How many recurring digits does each of them have?
  - Write a general rule connecting the numbers and the number of recurring digits.
  - Give a counter example to show that your rule does not work for all  $\frac{1}{p}$ , where  $p$  is a prime number.
- 7 Consider the numbers  $\frac{a}{7}$ , where  $a = \{1, 2, 3, 4, 5, 6\}$ .
- Write them as recurring decimals.
  - Describe the pattern that occurs.
- 8  $\frac{1}{7} = 0.142857142857\dots$  The repeated digits are 142857. Notice that  $142 + 857 = 999$ , which is a string of 9s.
- Show that the pattern occurs for  $\frac{1}{13}$ .
  - Find another example where the pattern works for  $\frac{1}{p}$ , where  $p$  is prime.
  - Find a counter example to show that  $p$  does not have to be prime for the pattern to work.

# 7.04 RATIONAL NUMBERS

We call the set of **rational numbers**  $\mathbf{Q}$ . It includes any number that can be written in the form  $\frac{a}{b}$ , where  $a$  and  $b$  are both integers and  $b$  is not equal to zero. This means that the set of integers  $\mathbf{Z}$ , terminating decimals and recurring decimals make up the set of rational numbers. Such examples are:

- 2, as it can be written as  $\frac{2}{1}$
- 0.875, as it can be written as  $\frac{7}{8}$
- $0.\bar{3}$ , as it can be written as  $\frac{1}{3}$

### IMPORTANT

A rational number is a number  $x$  that can be expressed in the form

$$x = \frac{a}{b}, \text{ where } a, b \in \mathbf{Z} \text{ and } b \neq 0.$$

$$\text{In set notation: } \mathbf{Q} = \left\{ x = \frac{a}{b} : a, b \in \mathbf{Z}, b \neq 0 \right\}.$$

Any rational number can be written in the form  $x = \frac{c}{d}$ , where  $c$  and  $d$  have no common factors except 1.  $d$  is usually taken to be positive, so for negative  $x$ ,  $c$  is negative.

### ○ Example 10

Show that each of the following numbers are rational by writing them in the form  $\frac{a}{b}$ , where  $a$  and  $b$  are integers and  $b$  is not equal to zero.

- a  $-3$             b  $0.75$             c  $2.17 \times 10^4$

#### Solution

- a Write  $-3$  as a fraction.

There are two equivalent forms. These are also equivalent to  $-\frac{3}{1}$ .

$$-3 = \frac{-3}{1} \text{ or } \frac{3}{-1}$$

- b Write  $0.75$  as a fraction.

Divide 75 by 100.

$\frac{3}{4}$  is the simplest form.

$$0.75 = \frac{75}{100} = \frac{3}{4}$$

- c Multiply  $2.17$  by  $10\,000$ .

Write the answer as a fraction.

$$\begin{aligned} 2.17 \times 10^4 &= 21\,700 \\ &= \frac{21\,700}{1} \end{aligned}$$

Example 11 is an example of a direct proof using rational numbers, showing that they are closed under addition.

### ○ Example 11

Prove that if  $x$  and  $y$  are rational numbers, then  $x + y$  is rational.

#### Solution

Represent  $x$  and  $y$  as a fraction.

Let  $x = \frac{a}{b}$  and  $y = \frac{c}{d}$ , where  $a, b, c$  and  $d$  are integers and  $b$  and  $d$  are not equal to zero.

Sum  $x$  and  $y$ .

$$\frac{a}{b} + \frac{c}{d}$$

Write the sum as a single fraction.

$$= \frac{ad + bc}{bd}$$

Explain why  $ad, bc$  and  $bd$  are all integers.

According to the Closure law of multiplication,  $ad, bc$  and  $bd$  are all integers.  $bd \neq 0$ .

Explain why  $ad + bc$  is an integer.

According to the Closure law of addition,  $ad + bc$  is an integer.

State the conclusion.

Hence  $\frac{ad + bc}{bd}$  is rational and can be written in the form  $\frac{e}{f}$  where  $e, f \in \mathbf{Z}, f \neq 0$ .

If  $x$  and  $y$  are rational numbers, then  $x + y$  is rational. **QED**

Example 12 is a proof using contradiction with rational numbers.

## ○ Example 12

Prove that there are no rational number solutions to the equation  $x^3 + x + 1 = 0$  using contradiction.

### Solution

Write the contradictory statement.

There is a rational number solution to  $x^3 + x + 1 = 0$ .

Substitute  $\frac{a}{b}$  into the equation, where  $a, b \in \mathbf{Z}$ ,  $b \neq 0$  and  $a$  and  $b$  have no common factors except 1, and  $b$  is positive.

$$\left(\frac{a}{b}\right)^3 + \frac{a}{b} + 1 = 0$$

Multiply both sides of the equation by  $b^3$ .

$$a^3 + ab^2 + b^3 = 0$$

Consider the different cases.

Zero is even.

1.  $a$  is odd and  $b$  is even.

$a^3$  is odd,  $ab^2$  is even,  $b^3$  is even.

LHS is odd and RHS is even: impossible.

2.  $a$  is even and  $b$  is odd.

$a^3$  is even,  $ab^2$  is even,  $b^3$  is odd.

LHS is odd and RHS is even: impossible.

3.  $a$  is odd and  $b$  is odd.

$a^3$  is odd,  $ab^2$  is odd,  $b^3$  is odd.

LHS is odd and RHS is even: impossible.

4.  $a$  is even and  $b$  is even.

$a$  and  $b$  would have a common factor of 2, which is a contradiction.

Write the answer.

There is no rational number solution to the equation  $x^3 + x + 1 = 0$ .

**QED**

### TI-Nspire CAS

Use **solve** to work out the solution to the equation with the calculator set to Exact calculation mode. A rational answer will be shown as a fraction or whole number. In this case the solution involves surds.

$$x = \frac{-6 \cdot (3 \cdot \sqrt{93} + 29)^{\frac{1}{3}}}{(2 \cdot (3 \cdot \sqrt{93} + 29))^{\frac{2}{3}} + 2 \cdot (3 \cdot \sqrt{93} + 29)^{\frac{1}{3}} + 2}$$

### ClassPad

Use the  $\sqrt[n]{x}$  application and set the calculator to **Standard**. A rational answer will be shown as a fraction or whole number. In this case the solution involves surds.

$$x = \frac{(12 \cdot \sqrt{93} - 108)^{\frac{1}{3}} - 2}{(12 \cdot \sqrt{93} - 108)^{\frac{1}{3}}}$$

## EXERCISE 7.04 Rational numbers

### Concepts and techniques

- 1 **Example 10** Show that each of the following numbers are rational by writing them in the form  $\frac{a}{b}$ , where  $a$  and  $b$  are integers and  $b$  is not equal to zero.
- a  $-\frac{2}{3}$                                   b 0.375                                  c  $5.67 \times 10^3$   
d  $8.2 \times 10^{-2}$                               e 0    f  $2\frac{3}{4}$
- 2 A recurring decimal belongs to which one of the following sets of numbers?  
A  $\mathbf{Z}^-$                                   B  $\mathbf{Z}^+$                                   C  $\mathbf{Z}$     D  $\mathbf{Q}$     E  $\mathbf{N}$
- 3 Simplify each of the following and show that the result is a rational number.
- a  $\frac{2}{3} + \frac{5}{7}$                               b  $2\frac{1}{2} - 3\frac{2}{5}$                               c  $\frac{2}{3} \times \frac{5}{7}$                               d  $2\frac{1}{2} \times 3\frac{2}{5}$                               e  $\frac{2}{3} \div \frac{5}{7}$

Use a direct proof to prove questions 4 to 6.

- 4 **Example 11** Show that the product of two rational numbers is rational.
- 5 Show that the difference between two rational numbers is rational.
- 6 a Show that if a rational number is divided by a rational number, then the result is a rational number.  
b Are the rational numbers closed under division?
- 7 Consider  $A = \left\{ \dots, \frac{1}{n}, \dots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2} \right\}$ , where  $n$  is a positive even integer.
- a Use a counter example to show that  $A$  is not closed under  
i addition                                  ii subtraction                                  iii division  
b Use a direct proof to show that  $A$  is closed under multiplication.

### Reasoning and communication

- 8 Use proof by contraposition to show that if  $x^2 - 2x + 5$  is not rational, then  $x$  is not rational.

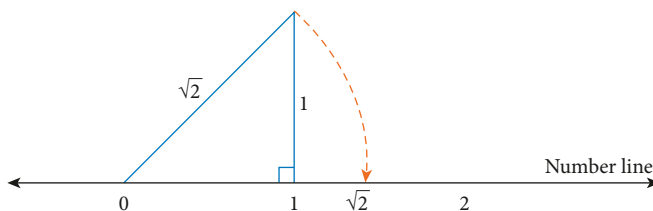
Use proof by contradiction to do questions 9 and 10.

- 9 **Example 12** Show that there are no rational solutions to the equation  $x^3 + 2x + 2 = 0$ .
- 10 Show that there are no rational solutions to the equation  $x^4 + x^2 + x - 2 = 0$ .
- 11 a Find an equation of the form  $ax^5 + bx^4 + cx^3 + dx^2 + ex + 1 = 0$  which has no rational solutions, where  $a, b, c, d, e \in \mathbf{Z}$ .  
b Sketch the graph of  $y = ax^5 + bx^4 + cx^3 + dx^2 + ex + 1$  for your values of  $a, b, c, d$  and  $e$ . Give the coordinates of any axis intercepts, correct to two decimal places.  
c Use proof by contradiction to show that  $ax^5 + bx^4 + cx^3 + dx^2 + ex + 1 = 0$  has no rational solutions using your values of  $a, b, c, d$  and  $e$ .

## 7.05 IRRATIONAL NUMBERS

Real numbers that cannot be written in the form  $\frac{a}{b}$ , where  $a, b \in \mathbf{Z}, b \neq 0$ , are called **irrational numbers**. Examples of such numbers are  $\sqrt{2}$ ,  $-3\sqrt{5}$ ,  $\log_2(5)$ ,  $e$  and  $\pi$ .  $\sqrt{4}$  is not an irrational number as it can be written as 2. Irrational numbers cannot be written as either terminating or recurring decimals. This means that decimal representations of these numbers will be approximations and not exact values; for example,  $\sqrt{2} \approx 1.414\ 213\ 562$  and  $e \approx 2.718\ 281\ 828$ . The ancient Babylonians used to approximate  $\pi$  as  $\frac{25}{8}$ .  $\pi$  has also been approximated as  $\frac{22}{7}$ ,  $\frac{333}{106}$  and 3.1416. However, it is actually an irrational number.

Irrational numbers that must be expressed in the form  $\sqrt[n]{a}$ , where  $n, a \in \mathbf{N}$ , are called **surds**. Even though it is not possible to find an exact decimal value for a surd, each surd corresponds to some exact value. This can be shown by the construction on the right.



Irrational numbers that are not surds or combinations of surds are called **transcendental numbers**; for example,  $e$  and  $\pi$ . These numbers can also be represented on a number line.

### IMPORTANT

#### Irrational numbers

An irrational number cannot be expressed in the form  $\frac{a}{b}$ ,  $b \neq 0$ , where  $a$  and  $b$  are integers.

You can also choose  $b > 0$  instead because  $a$  positive or negative is enough.

Expressed as decimals, irrational numbers do not recur and do not terminate. However, they do have definite locations on the number line. In set notation:  $\mathbf{Q}' = \{x \in \mathbf{R}: x \notin \mathbf{Q}\}$ .

In Examples 13 and 14, proof by contradiction is used to show that  $\sqrt{2}$  and  $\log_2(5)$  are irrational. In both cases, you use the fact that any number in fractional form can always be written in its simplest form by cancelling down any factors other than 1.

#### ○ Example 13

Use proof by contradiction to show that  $\sqrt{2}$  is an irrational number.

#### Solution

Write the contradictory statement.

Assume that  $\sqrt{2}$  is a rational number.

What does this mean?

Then  $\sqrt{2}$  can be written as a ratio of integers.

Write  $\sqrt{2}$  as a rational number in simplest form.

Write  $\sqrt{2} = \frac{a}{b}$ ,  $a, b \in \mathbf{Z}, b \neq 0$ , where  $a$  and  $b$  only have 1 as a common factor.



Rational and irrational numbers



Square both sides of the equation.

$$2 = \frac{a^2}{b^2}$$

Rearrange the equation.

$$a^2 = 2b^2$$

What does  $a^2 = 2b^2$  mean?

$a^2$  must be even, so  $a$  must be even.

Write  $a$  as double another integer.

$$a = 2k, \text{ where } k \in \mathbf{Z}$$

Substitute  $a = 2k$  into the equation.

$$(2k)^2 = 2b^2$$

Simplify and make  $b^2$  the subject.

$$2b^2 = 4k^2$$

$$b^2 = 2k^2$$

What does  $b^2 = 2k^2$  mean?

$b^2$  must be even, so  $b$  is even.

State the critical point.

$a$  and  $b$  are even, so they have a common factor of 2.

Write the contradiction.

This contradicts the assumption that  $\sqrt{2}$  is rational, so it must be false.

Write the conclusion.

$\sqrt{2}$  is irrational.

**QED**

### ○ Example 14

Use proof by contradiction to show that  $\log_2(5)$  is irrational.

#### Solution

Write the contradictory statement.

Assume that  $\log_2(5)$  is a rational number.

Write  $\log_2(5)$  as a rational number in simplest form.

Write  $\log_2(5) = \frac{a}{b}$ ,  $a, b \in \mathbf{Z}^+$ , where  $a$  and  $b$  only have 1 as a common factor.

Put in index form.

$$2^{\frac{a}{b}} = 5$$

Raise both sides of the equation to the power of  $b$ .

$$\left(2^{\frac{a}{b}}\right)^b = 5^b$$

Tidy up the equation.

$$2^a = 5^b$$

State the critical point.

For any positive integer  $a$ ,  $2^a$  is even  
For any positive integer  $b$ ,  $5^b$  is odd  
A positive integer cannot be both even and odd.

Write the contradiction.

This contradicts the assumption that  $\log_2(5)$  is a rational number, so it must be false.

Write the conclusion.

$\log_2(5)$  is irrational.

**QED**

## INVESTIGATION Infinite continued fractions for irrational numbers

This investigation is designed for a computer-based CAS such as Mathematica.

Irrational numbers can be written as Infinite continued fractions. These are fractions in the

form  $a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$ , where  $a_0 \in \mathbf{Z}$  and  $a_1, a_2, \dots$  are positive integers.

a  $\sqrt{2}$  can be written as an infinite continued fraction,  $1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}$ .

Write the decimal equivalent for each of the following finite continued fractions and comment on the result.

$$1 + \frac{1}{2}, 1 + \frac{1}{2 + \frac{1}{2}}, 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}, 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}}$$

The decimal equivalent of other irrational numbers can be constructed in the same way.

- b You can construct the decimal 0.01001000100001... using an infinite continued fraction.
- i Find the finite continued fractions for 0.01 and 0.01001 by finding values for  $a_0, a_1, a_2$  and  $a_3$ .

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2}} = 0.01, \quad a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3}}} = 0.01001$$

- ii Using an appropriate CAS, find the continued fraction for 0.010010001. The command for Mathematica is `ContinuedFraction[0.010010001]`. This cannot presently be done on CAS calculators.
- c Find infinite continued fractions for  $e$  and  $\pi$ . The command for Mathematica is `ContinuedFraction[Pi,20]`. Comment on the results.
- d Watch the Mathematica demonstration at the website.

<http://demonstrations.wolfram.com/ContinuedFractions/>

What conclusion can you draw about square root functions as opposed to cube roots or higher? Give some examples to support your conclusion.



WolframAlpha

Infinite continued fractions can be used to construct decimals like 0.01001000100001... as shown in the investigation. However, there is a much simpler way to construct this decimal, as shown in Example 15.

### ○ Example 15

Construct the decimal 0.01001000100001... by adding fractions of the form  $\frac{1}{10^{2+a}}$ , where  $a \in \mathbf{Z}^+ \cup \{0\}$ .

#### Solution

Write 0.01 as a fraction.

$$\frac{1}{100} = \frac{1}{10^2}$$

Write 0.000 01 as a fraction.

$$\frac{1}{100\ 000} = \frac{1}{10^5}$$

Write 0.000 000 001 as a fraction.

$$\frac{1}{1\ 000\ 000\ 000} = \frac{1}{10^9}$$

Write 0.01001000100001... as the sum of the fractions by observing the pattern.

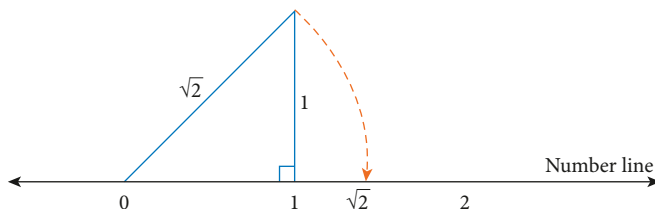
$$\frac{1}{10^2} + \frac{1}{10^5} + \frac{1}{10^9} + \frac{1}{10^{14}} + \dots + \frac{1}{10^{\frac{1}{2}n^2 + \frac{3}{2}n}} + \dots$$

$n \in \mathbf{N}$

## EXERCISE 7.05 Irrational numbers

### Concepts and techniques

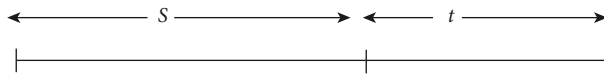
- Which one of the following is not an irrational number?  
 A  $2\sqrt{6}$       B  $\pi$       C  $e$       D  $-\sqrt{99}$       E  $2\sqrt{16}$
- Give a counter example to show that irrational numbers are not closed under  
 a addition      b subtraction      c multiplication      d division.
- Using the result  $1^2 + 2^2 = 5$ , show  $\sqrt{5}$  on the number line below.
- Does  $\pi = \frac{22}{7}$ ? Explain.
- Besides  $\pi$  and  $e$ , give two other examples of transcendental numbers.



### Reasoning and communication

- Example 13** Use proof by contradiction to show that  $\sqrt{5}$  is irrational.
- Example 14** Use proof by contradiction to show that  $\log_2(3)$  is irrational.
- Using simultaneous equations or finite differences, show that the sequence 2, 5, 9, 14, ... in Example 15 can be written as  $\frac{1}{2}n^2 + \frac{3}{2}n$ .

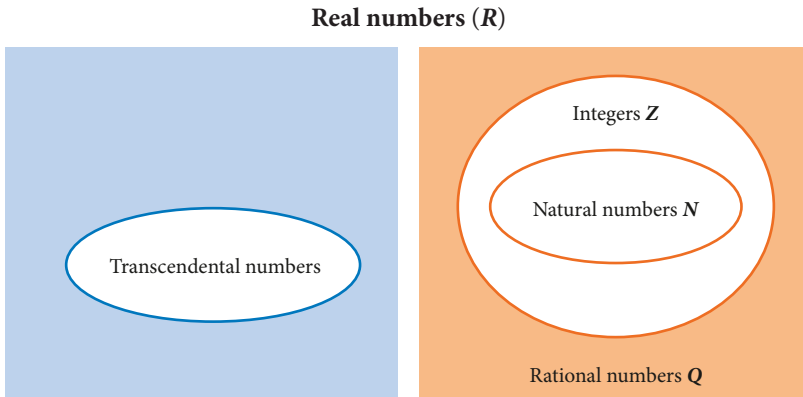
- 9 **Example 15** Construct the decimal  $0.01000100001000001\dots$  by adding fractions of the form  $\frac{1}{10^{2+a}}$ , where  $a \in \mathbf{Z}^+ \cup \{0\}$ .
- 10 Construct the decimal  $0.01002000400008\dots$  by adding fractions of the form  $\frac{2^b}{10^{2+a}}$ , where  $a, b \in \mathbf{Z}^+ \cup \{0\}$ .
- 11 Give the conditions for which the solutions to the quadratic equation  $ax^2 + bx + c = 0$ , where  $a, b, c \in \mathbf{Z}$ , are irrational.
- 12 Give an example of a cubic equation of the form  $ax^3 + bx^2 + cx + d = 0$  which only has irrational solutions.
- 13 a Solve  $x^2 - x - 1 = 0$ .
- b Use the result  $\frac{s+t}{s} = \frac{s}{t} = 1.618\dots = \varphi = \mathbf{Golden Ratio}$ , where  $s > t$ , to answer the following questions.



- i Find the Golden Ratio in surd form. Hint: let  $t = 1$ .
- ii Prove by contradiction that the Golden Ratio is irrational.

## 7.06 REAL NUMBERS

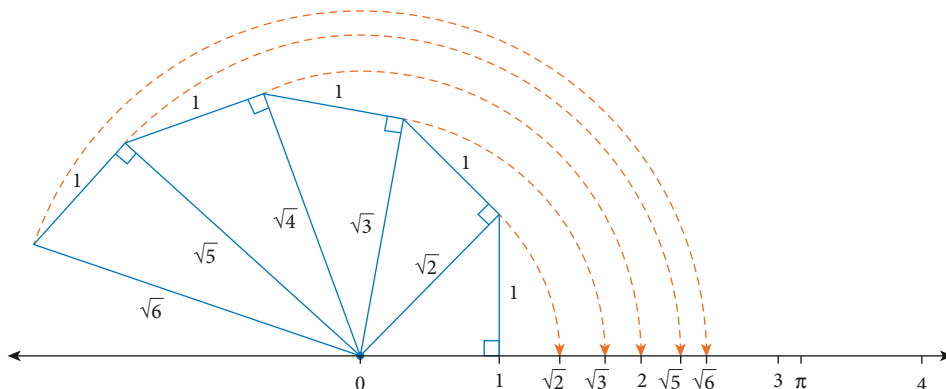
The inclusion of irrational numbers with rational numbers gives the set of **real numbers**,  $\mathbf{R}$ . The diagram below shows the relationship between the various sets of numbers we have discussed in this chapter.



The set of real numbers can be thought of as the universal set, where

$$\mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q} \subset \mathbf{R} \text{ and } \{\text{irrational numbers}, \mathbf{Q}'\} \subset \mathbf{R}.$$

The set of real numbers can be represented on a number line as shown in the diagram below.



Examples 16 and 17 are proofs by contradiction which involve real numbers.

### ○ Example 16

Use proof by contradiction to show that for all real numbers  $x$  and  $y$ , if  $x$  is rational and  $y$  is irrational, then  $x + y$  is irrational.

#### Solution

Write the contradictory statement.

Assume that there exists a rational number  $x$  and an irrational number  $y$  such that  $x + y$  is rational.

Write  $x$  as a rational number.

$$x = \frac{a}{b}, \text{ where } a, b \in \mathbf{Z}, b \neq 0$$

Write  $x + y$  as rational number.

$$\frac{a}{b} + y = \frac{p}{q}, \text{ where } a, b, p, q \in \mathbf{Z}, \text{ and } b, q \neq 0$$

Make  $y$  the subject.

$$y = \frac{p}{q} - \frac{a}{b}$$

Show that  $y = \frac{p}{q} - \frac{a}{b}$  is rational.

$$\frac{p}{q} - \frac{a}{b} = \frac{bp - aq}{bq} = \frac{m}{n}, \text{ where } m, n \in \mathbf{Z}, n \neq 0$$

State the vital point.

Since  $\frac{p}{q} - \frac{a}{b}$  is rational,  $y$  is rational.

State the contradiction.

But this is a contradiction, as  $y$  is irrational.

State the effect of the contradiction.

Thus, there is no rational number  $x$  and irrational number  $y$  such that  $x + y$  is rational.

State the conclusion.

For all real numbers  $x$  and  $y$ , if  $x$  is rational and  $y$  is irrational, then  $x + y$  is irrational. **QED**

### ○ Example 17

Prove  $\frac{1}{x+y} \neq \frac{1}{x} + \frac{1}{y}$  by contradiction for all non-zero real numbers  $x$  and  $y$ .

#### Solution

Write the contradictory statement.

Assume that there exist  $x, y \in \mathbf{R}$ ,  
where  $x, y \neq 0$ , such that  $\frac{1}{x+y} = \frac{1}{x} + \frac{1}{y}$ .

Write the RHS as a single fraction.

$$\text{Then } \frac{1}{x+y} = \frac{y+x}{xy}.$$

Cross multiply.

$$\text{So } (y+x)^2 = xy$$

Expand the LHS.

$$y^2 + 2xy + x^2 = xy$$

Simplify.

$$\text{So } x^2 + xy + y^2 = 0$$

Use the quadratic formula.

$$\text{This gives } x = \frac{-y \pm \sqrt{y^2 - 4 \times 1 \times y^2}}{2}$$

Simplify.

$$\text{So } x = \frac{-y + \sqrt{-3y^2}}{2} \text{ or } x = \frac{-y - \sqrt{-3y^2}}{2}$$

State the vital point.

But for  $y \in \mathbf{R}$   $\sqrt{-3y^2}$  is not real, so  $x \notin \mathbf{R}$ .

State the contradiction.

But this contradicts the assumption that  $x \in \mathbf{R}$ .

State the effect of the contradiction.

Thus there are no  $x, y \in \mathbf{R}$ , where  $x, y \neq 0$ ,  
such that  $\frac{1}{x+y} = \frac{1}{x} + \frac{1}{y}$ .

Write the answer.

For all nonzero real numbers  $x$  and  $y$ ,  
 $\frac{1}{x+y} \neq \frac{1}{x} + \frac{1}{y}$ .

**QED**

Quadratic equations,  $y = ax^2 + bx + c$ , where  $a, b, c \in \mathbf{R}$ , have real roots if the discriminant,  $b^2 - 4ac$  is greater than or equal to zero.

## Example 18

For what values of  $c$  will the solutions of the quadratic equation  $x^2 + x + c = 0$  be:

a real

b integers?

### Solution

a Solve  $x^2 + x + c = 0$  for  $x$  using the quadratic formula.

$$x = \frac{-1 \pm \sqrt{1-4c}}{2}$$

For real solutions the discriminant has to be greater than or equal to zero.

$$1 - 4c \geq 0$$

Make  $c$  the subject.

$$-4c \geq -1$$

$$c \leq \frac{1}{4}$$

b Write the solutions.

$$x = \frac{-1 \pm \sqrt{1-4c}}{2}$$

Choose integer solutions.

$$\text{Put } \frac{-1 + \sqrt{1-4c}}{2} = m \text{ and } \frac{-1 - \sqrt{1-4c}}{2} = n; n, m \in \mathbf{Z}.$$

Choose one value and square it.

$$\text{This gives } 1 - 4c = (2m + 1)^2.$$

Rearrange to make  $c$  the subject.

$$c = \frac{1 - (2m + 1)^2}{4} = \frac{-4m^2 - 4m}{4} = -m^2 - m = -m(m + 1)$$

Comment for other value.

$c$  has the same form for  $n$ .

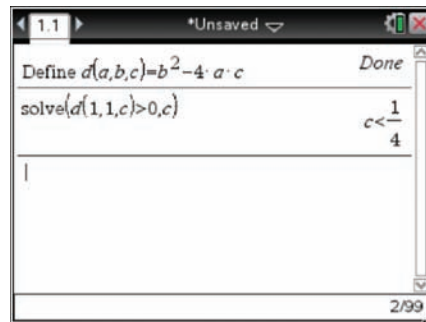
Write the answer.

For integer solutions,  $c$  must be of the form  $-m(m + 1)$ , where  $m \in \mathbf{Z}$ .

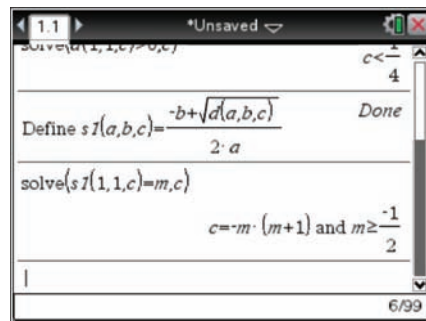
### a TI-Nspire CAS

Define the function  $d(a, b, c)$  as  $d(a, b, c) = b^2 - 4ac$  for the discriminant. Make sure that you put a multiplication between  $a$  and  $c$ .

Now solve  $d(1, 1, c) > 0$  for  $c$ .



b Choose one of the solutions, say  $s1 = (-b + \sqrt{d(a,b,c)}) / (2a)$  and solve  $s1 = m$  for  $c$ .

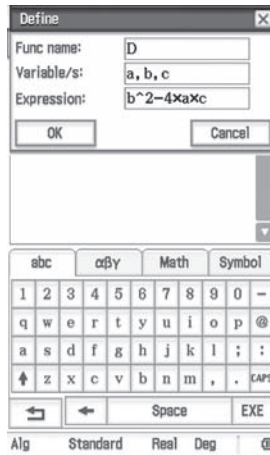


**a ClassPad**

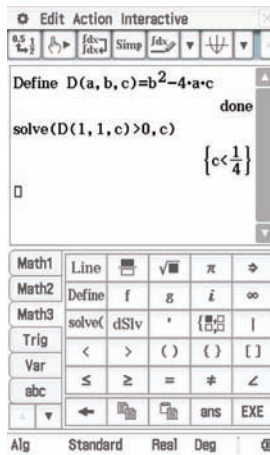
Use the  $\sqrt{\quad}$  application and set the calculator to **Standard**.

Define  $D(a, b, c) = b^2 - 4ac$ .

(The **Define** function is found at the bottom of the **Interactive** menu.) Fill in the screen so that it matches the one on the right and tap **OK**.



In this case,  $a = 1$ ,  $b = 1$  and  $c$  is  $c$ . Solve for  $D > 0$ , looking for values of  $c$ .

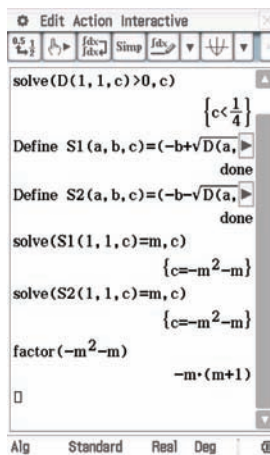


**b** Define the two solutions as  $S1$  and  $S2$  (see right screen).

$$S1 = \frac{-b + \sqrt{D(a, b, c)}}{2a}$$

$$S2 = \frac{-b - \sqrt{D(a, b, c)}}{2a}$$

Solve  $S1 = m$  for  $c$  and also  $S2 = m$  for  $c$ .



Write the solutions.

a  $c \leq \frac{1}{4}$

b  $c = -m(m + 1)$  for  $m \in \mathbb{Z}$



## EXERCISE 7.06 Real numbers

### Concepts and techniques

- 1 a If  $x = 2 + \sqrt{3}$  and  $y = 2 - \sqrt{3}$ , then find each of the following.
- i  $x + y$       ii  $x - y$       iii  $xy$       iv  $\frac{x}{y}$
- b State which subset of the real number system each of the results in part a belongs to.
- 2 Is the set of real numbers closed under
- a addition?      b subtraction?      c multiplication?      d division?
- 3 If  $x = 3$  and  $y = 6$ , find each of the following.
- a  $\frac{1}{x+y}$       b  $\frac{1}{x} + \frac{1}{y}$       c  $\frac{1}{x-y}$       d  $\frac{1}{x} - \frac{1}{y}$

### Reasoning and communication

For questions 4 to 7, use proof by contradiction to show that each of the statements is true.

- 4 **Example 16** For all real numbers  $x$  and  $y$ , if  $x$  is rational and  $y$  is irrational, then  $x \times y$  is irrational.
- 5 For all real numbers  $x$  and  $y$ , if  $x$  is rational and  $y$  is irrational, then  $x - y$  is irrational.
- 6 **Example 17** For all nonzero real numbers  $x$  and  $y$ ,  $\frac{1}{x-y} \neq \frac{1}{x} - \frac{1}{y}$ .
- 7 a In a right-angled triangle, the length of the hypotenuse is less than the sum of the lengths of the other two sides.  
b Give an example of where the solutions to  $a^2 + b^2 = c^2$  are integer values only.
- 8 **Example 18** Consider the quadratic equation  $x^2 + x + c = 0$ . For what values of  $c$  will the solutions be rational?
- 9 Give a counter example to show that if the solutions to the equation
- $$x^3 + bx^2 + cx + d = (x - e)(x - f)(x - g) = 0$$
- are irrational, then  $d$  is irrational is a false statement.

## 7.07 THE PRINCIPLE OF MATHEMATICAL INDUCTION

**Mathematical induction** is a technique which is used to prove whether a generalisation is true or not.

Proof by induction follows a set of formal steps.

### IMPORTANT

#### The principle of mathematical induction

Let there be associated with each positive integer  $n$ , a proposition  $P(n)$ .

If 1  $P(1)$  is true, and

2 for all  $k$ , if  $P(k)$  is true, it follows that  $P(k + 1)$  is true,

then  $P(n)$  is true for all positive integers  $n$ .

## IMPORTANT

To prove something by mathematical induction, you should follow the steps below.

Step 1: Under **RTP** (required to prove) state what has to be proved.

Step 2: Prove that the statement is true for  $n = 1$ .

Step 3: Assume that the statement is true for  $n = k$ , where  $k \in \mathbf{Z}^+$ .

Step 4: Show that it necessarily follows that the statement is true for  $n = k + 1$ .

Step 5: Make a formal statement that 'By the principle of mathematical induction ...' and put in the statement, followed by **QED**.

Example 19 is a simple example of proving results for sums. Example 20 is more complicated.

### ○ Example 19

Use the method of mathematical induction to prove that

$$1 + 3 + 5 + 7 + 9 + \dots + (2n - 1) = n^2 \text{ for all } n \in \mathbf{Z}^+.$$

#### Solution

State what has to be proved.

**RTP**

$$1 + 3 + 5 + 7 + 9 + \dots + (2n - 1) = n^2 \text{ for all } n \in \mathbf{Z}^+.$$

Show that it is true for  $n = 1$ .

$$\text{LHS} = 1$$

$$\text{RHS} = 1^2 = 1$$

State the first part.

Since the LHS = RHS, it is true for  $n = 1$ .

Assume that it is true for  $n = k$ , where  $k \in \mathbf{Z}^+$ .

Assume that  $1 + 3 + 5 + 7 + 9 + \dots + (2k - 1) = k^2$  for  $k \in \mathbf{Z}^+$

Add the  $(k + 1)$ th term to both sides of the equation.

$$1 + 3 + 5 + 7 + 9 + \dots + (2k - 1) + [2(k + 1) - 1] \\ = k^2 + [2(k + 1) - 1]$$

Expand the RHS and simplify.

$$= k^2 + 2k + 2 - 1 \\ = k^2 + 2k + 1$$

Factorise so that the RHS is in the same form as the original statement.

$$= (k + 1)^2$$

Summarise the statement for  $k + 1$ .

Thus  $1 + 3 + 5 + 7 + 9 + \dots + (2k - 1) + [2(k + 1) - 1] = (k + 1)^2$  so it is true for  $k + 1$ .

Make a formal statement for all values of  $n$ .

By the principle of mathematical induction,

$$1 + 3 + 5 + 7 + 9 + \dots + (2n - 1) = n^2 \text{ for all } n \in \mathbf{Z}^+.$$

Replace  $k + 1$  with  $n$ .

**QED**

## ○ Example 20

Use the method of mathematical induction to prove that

$$1 + 4 + 9 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6} \text{ for all } n \in \mathbf{Z}^+.$$

### Solution

State what has to be proved.

**RTP**

$$1 + 4 + 9 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6} \text{ for all } n \in \mathbf{Z}^+.$$

Show that it is true for  $n = 1$ .

$$\text{LHS} = 1^2 = 1$$

$$\text{RHS} = \frac{1 \times 2 \times 3}{6} = \frac{6}{6} = 1$$

State the first part.

Since the LHS = RHS, it is true for  $n = 1$ .

Assume that it is true for  $n = k$ ,  
where  $k \in \mathbf{Z}^+$ .

$$\text{Assume that } 1 + 4 + 9 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6} \text{ for } k \in \mathbf{Z}^+$$

Add the  $(k + 1)$ th term to both sides  
of the equation.

$$1 + 4 + 9 + \dots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

Simplify the RHS by using the LCD  
of 6.

$$\text{RHS} = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

Use  $k + 1$  as a common factor.

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$

Expand  $[k(2k + 1) + 6(k + 1)]$ .

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

Factorise  $(2k^2 + 7k + 6)$ .

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

Summarise the statement for  $k + 1$ .

$$= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6} \text{ so it is true for } k+1.$$

Make a formal statement.

By the principle of mathematical induction,

Replace  $k + 1$  with  $n$ .

$$1 + 4 + 9 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6} \text{ for all } n \in \mathbf{Z}^+. \quad \text{QED}$$

Example 21 involves proving a divisibility result using mathematical induction.

### ○ Example 21

Use the method of mathematical induction to prove that  $3^{2n+4} - 2^{2n}$  is divisible by 5 for any positive integer  $n$ .

#### Solution

State what has to be proved.

**RTP**

$3^{2n+4} - 2^{2n}$  is divisible by 5 for any positive integer  $n$ .

Show that it is true for  $n = 1$ .

$$3^{2n+4} - 2^{2n} = 3^6 - 2^4 = 725$$

State the first part.

725 is divisible by 5.

Assume that it is true for  $n = k$ , where  $k \in \mathbb{Z}^+$ .

Write  $3^{2k+4} - 2^{2k} = 5M$ , where  $M \in \mathbb{Z}^+$ .

Write the expression for  $k + 1$ .

$$3^{2(k+1)+4} - 2^{2(k+1)}$$

Simplify

$$= 3^{2k+4+2} - 2^{2k+2}$$

Write the expression in terms of  $3^{2k+4} - 2^{2k}$ .

$$\begin{aligned} &= 3^{2k+4} \times 9 - 2^{2k} \times 4 \\ &= 3^{2k+4} \times 5 + 3^{2k+4} \times 4 - 2^{2k} \times 4 \\ &= 4(3^{2k+4} - 2^{2k}) + 5 \times 3^{2k+4} \end{aligned}$$

Substitute  $5M$  for  $3^{2k+4} - 2^{2k}$ .

$$= 4 \times 5M + 5 \times 3^{2k+4}$$

Factorise, since 5 is a common factor.

$$= 5(4M + 3^{2k+4}), \text{ which is divisible by 5}$$

Summarise the statement for  $k + 1$ .

$3^{2(k+1)+4} - 2^{2(k+1)}$  is divisible by 5 so it is true for  $k + 1$ .

Make a formal statement.

By the principle of mathematical induction,

Replace  $k + 1$  with  $n$ .

$3^{2n+4} - 2^{2n}$  is divisible by 5 for any positive integer  $n$ .

**QED**

## EXERCISE 7.07 The principle of mathematical induction

### Reasoning and communication

1 **Example 19** Use mathematical induction to prove each of the following for  $n \in \mathbf{Z}^+$ .

- a  $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$  for all  $n \geq 1$ .
- b  $1 + 4 + 7 + 10 + \dots + (3n - 2) = \frac{n(3n-1)}{2}$  for all  $n \geq 1$ .
- c  $2 + 4 + 6 + \dots + 2n = n(n + 1)$  for all  $n \geq 1$ .

2 **Example 20** Show that for every positive integer  $n$ ,

- a  $1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{n(4n^2 - 1)}{3}$
- b  $2 + 2^2 + 2^3 + 2^4 + \dots + 2^n = 2^{n+1} - 2$
- c  $1^3 + 3^3 + 5^3 + \dots$  to  $n$  terms  $= n^2(2n^2 - 1)$

3 Use mathematical induction to prove each of the following, for all positive integers  $n \geq 1$ .

- a  $\frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \frac{1}{7 \times 9} + \dots$  to  $n$  terms  $= \frac{n}{3(2n+3)}$
- b  $\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) \dots \left(1 - \frac{1}{n}\right) = \frac{1}{n}$

4 **Example 21** Prove the following, where  $n \in \mathbf{Z}^+$ .

- a  $3^{2n} - 1$  is divisible by 8.
- b  $4^n - 1$  is divisible by 3.
- c  $3^n - 1$  is divisible by 2.
- d  $n^7 - n$  is divisible by 7.
- e  $n(n + 1)(n + 2)$  is divisible by 6.

5 Prove the following inequalities using the method of mathematical induction.

- a  $2^n > n$  for all  $n \in \mathbf{Z}^+$ .
- b  $\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{n+n} > \frac{13}{24}$ , for integers  $n > 1$ .

6 If  $r \neq 1$ , show that, for any positive integer  $n$ ,  $a + ar + ar^2 + \dots + ar^n = \frac{a(r^n - 1)}{r - 1}$ .

# CHAPTER SUMMARY

## REAL NUMBERS AND PROOFS



- The **real** number system,  $\mathbf{R}$ , consists of the **rational** numbers,  $\mathbf{Q}$  and the **irrational** numbers,  $\mathbf{Q}'$ .
- Any rational number can be written in the form  $\frac{a}{b}$ , where  $a, b \in \mathbf{Z}$  and  $b \neq 0$ . They can be written as **terminating** decimals or **recurring** decimals.
- An **integer** is a positive or negative whole number.  $\mathbf{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ .
- The counting numbers or **natural numbers**,  $\mathbf{N}$  is the set of positive integers.  $\mathbf{N} = \{1, 2, 3, \dots\}$ .
- The set of integers can be divided into **even** numbers,  $\{\dots, -4, -2, 0, 2, 4, \dots\}$  and **odd** numbers  $\{\dots, -3, -1, 1, 3, \dots\}$ . Even numbers can be written in the form  $2k$  and odd numbers in the form  $2k + 1$ , where  $k$  is an integer.
- The set of irrational numbers consists of **surds** and **transcendental numbers**. They cannot be written in the form  $\frac{a}{b}$ , where  $a, b \in \mathbf{Z}$  and  $b \neq 0$ . They do not have digits that repeat when represented as a decimal.
- Surds are irrational numbers that can be expressed in the form  $\sqrt[n]{a}$ , where  $n, a \in \mathbf{N}$ .
- **Transcendental numbers** are irrational numbers that cannot be expressed as a combination of surds; for example,  $e$  and  $\pi$ .
- A **perfect** number is a positive integer that is equal to the sum of its positive **proper divisors**. Proper divisors are the factors of the number, excluding the number itself.
- A **deficient** number is a positive integer where the sum of its proper divisors is less than the number itself.
- An **abundant** number is a positive integer where the sum of its proper divisors is greater than the number itself.
- **Amicable** numbers are pairs of positive integers where each one is the sum of the proper divisors of the other.
- A **prime number** is a positive integer that has exactly two factors, one and itself. A **composite** number is a positive integer that has more than two factors.
- The **Sieve of Eratosthenes** is a method for finding prime numbers up to a particular number,  $n$ . Only the prime numbers less than or equal to  $\sqrt{n}$  need to be checked.
- **Double or twin** primes differ by two. A **Mersenne** prime is a prime number that can be written in the form  $2^p - 1$ , where  $p$  is prime.
- A set is **closed** under an operation  $*$  if and only if, for all  $a$  and  $b$  in  $S$ ,  $a * b$  is also in  $S$ .
- A **direct** proof involves giving logical arguments to show that a statement is true.
- A **counterexample** shows that a statement is false by giving one instance where it is not true.
- **Contraposition** means that 'if  $p$  then  $q$ ' is the same as 'if not  $q$  then not  $p$ '.
- In a proof by **contradiction** the statement you are trying to prove, say  $A$ , is assumed to be false. This is then shown to lead to something that contradicts the assumption. This means that the assumption is false, so  $A$  is true.

- The principle of mathematical induction states that for a statement  $P(n)$ , if  $P(1)$  is true and for all  $k$ , if  $P(k)$  is true then it follows that  $P(k + 1)$  is true then  $P(n)$  is true for all positive integers  $n$ .
- To prove something by **mathematical induction**, perform the following steps.
  - Step 1: Under **RTP** (required to prove) state what has to be proved.
  - Step 2: Prove that the statement is true for  $n = 1$ .
  - Step 3: Assume that the statement is true for  $n = k$ , where  $k \in \mathbb{Z}^+$ .
  - Step 4: Show that it necessarily follows that the statement is true for  $n = k + 1$ .
  - Step 5: Make a formal statement that ‘By the principle of mathematical induction ...’ and put in the statement, followed by **QED**.

# CHAPTER REVIEW

## REAL NUMBERS AND PROOFS



### Multiple choice

- 1 **Example 1** Which one of the following sets of natural numbers does not contain an even number?  
A composite  
B prime  
C numbers of the form  $n^2 + n + 1$ , where  $n$  is natural  
D numbers of the form  $n^2 + 2n + 3$ , where  $n$  is natural  
E perfect squares
- 2 **Example 16** Which one of the following is incorrect?  
A  $Q' \subset R$                       B  $N \subset Z^+ \subset Q \subset R$                       C  $N \subset Z \subset Q \subset R$   
D  $Q \subset Q' \subset R$                       E  $N \subset Z^+ \subset R^+$
- 3 **Example 7, 8** If  $a$  is the last digit of the number  $219a$ , then  $\frac{219a}{175}$  can be written as a terminating decimal if  $a$  equals  
A 1 only                      B 1 and 8 only                      C 8 only                      D 2 and 8 only                      E 0 and 1 only
- 4 **Example 4** The set of integers is not closed under  
A multiplication                      B squaring                      C division                      D subtraction                      E addition
- 5 **Example 9**  $\frac{1}{2}$  does not equal  
A 0.500...                      B  $0.4\bar{9}$                       C 0.5                      D 0.499                      E  $\frac{5}{10}$

### Short answer

- 6 **Example 1** a Show that when three odd numbers are added together, the result is an odd number.  
b Show that when three odd numbers are multiplied together, the result is an odd number.
- 7 **Example 2** Show that 100 is not a perfect number.
- 8 **Example 3** a Write 511 in the form  $2^a - 1$ , where  $a$  is an integer, and explain why it is not a Mersenne prime.  
b By testing prime numbers less than  $\sqrt{511}$ , show that 511 is not a prime number.
- 9 **Example 4** Give counter examples to show that each of the following statements is false.  
a When an even number is divided by an even number, the result is an even number.  
b The set of composite numbers is closed under subtraction.
- 10 **Example 5** Prove by contraposition that 2 is the only even prime number.





## Application



Practice quiz

- 27 a i Write out the factors of 496.  
 ii Find the sum of the five smallest factors and comment on the result.  
 iii Find the sums of the six smallest factors and comment on the result.  
 iv Repeat the process until there are no more factors to sum.  
 v What type of number is 496?
- b i Write 496 in the form  $2^{p-1}(2^p - 1)$ .  
 ii What does  $2^{p-1}$  equal?  
 iii What does  $2^p - 1$  equal?
- c Use a direct proof to show that  $2^p - 1$  is equal to the sum of the factors of  $2^{p-1}$ , where  $p$  is a positive integer.
- d i Write each of the following as the product of two factors, one of which is  $x - 1$ .  
 $x^2 - 1$ ,  $x^3 - 1$ ,  $x^4 - 1$ ,  $x^5 - 1$  and  $x^n - 1$ , where  $n$  is a positive integer greater than 1.  
 ii Hence use proof by contraposition to show that if  $2^p - 1$  is prime, then  $p$  is prime.  
 iii Give a counter example to show that if  $p$  is prime, then  $2^p - 1$  is prime is a false statement.